Spin-momentum correlation in relativistic single particle quantum states

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Abstract

This paper was concerned with the spin-momentum correlation in single-particle quantum states, which is described by the mixed states under Lorentz transformations. For convenience, instead of using the superposition of momenta we use only two momentum eigen states (p_1 and p_2) that are perpendicular to the Lorentz boost direction. Consequently, in 2D momentum subspace we show that the entanglement of spin-momentum in the moving frame depends on the angle between them. Therefore, when spin and momentum are perpendicular the measure of entanglement is not observer-dependent quantity in inertial frame. Likewise, we have calculated the measure of entanglement (by using the concurrence) and has shown that entanglement decreases with respect to the increasing of observer velocity. Finally, we argue that, Wigner rotation is induced by Lorentz transformations can be realized as controlling operator.

Keywords : Spin-momentum correlation, Relativistic entanglement, Quantum gate PACS numbers: 03.67.Hk, 03.65.Ta

1 Introduction

In two recent decades quantum entanglement has become as one of the most important resources in the rapidly growing field of quantum information processing with remarkable applications on it [1], and was based on the fact that the existence of entangled states produces nonclassical phenomena. Therefore, specifying that a particular quantum state is entangled or separable is important because if the quantum state be separable then its statistical properties can be explained entirely by classical statistics.

Relativistic aspects of quantum mechanics have recently attracted much attention in the context of the theory of quantum information, especially on quantum entanglement[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Peres *et* al.[6] have recently observed that the reduced spin density matrix of a single spin-$\frac{1}{2}$ particle is not a relativistic invariant, and

Wigner rotations correlate spin with the particle momentum distribution when is observed in a moving frame [7]. Gingrich and Adami have shown that the entanglement between the spins of two particles is carried over to the entanglement between the momenta of the particles by the Wigner rotation, even though the entanglement of the entire system is Lorentz invariant [8]. Terashimo and Ueda [9] and Czarchor [10] suggested that the degree of violation of the Bell inequality depends on the velocity of the pair of spin-$\frac{1}{2}$ particles or the observer with respect to the laboratory. Alsing and Milburn studied the Lorentz invariance of entanglement and showed that the entanglement fidelity of the bipartite state is preserved explicitly. Instead of state vector in the Hilbert space, they have used a 4-component Dirac spinor or a polarization vector in favor of quantum field theory [11]. Ahn also calculated the degree of violation of the Bell's inequality which is decreases with increasing of velocity of the observer [12]. Most of the previous works were concerned with the pure states although authors in [13, 14, 15] have considered mixed quantum states that are described by superposition of momenta with Gaussian distribution, where Lorentz transformation introduces a transfer of entanglement between different degrees of freedom. While the entanglement between spins and momentums of particles may change, separately. However, the total entanglement of particle-particle is the same in all inertial frames. Beside of the pervious works that are concerned of study on the entanglement between quantum states of two particles, here we generalize this to the spin-momentum correlation of relativistic single-particle (by using the concurrence) and show that the measure of entanglement depends on the angle between spin and momentum and it decreases with increasing of velocity of the observer. Also it has been shown that the Wigner angle depends on momentum, so Wigner rotation behaves as a quantum gate or controlling operators. Thus using this quantum gate the spin-momentum entanglement changes in the framework of special relativity.

This paper is organized as follows: Sec. II, is devoted to single-particle relativistic quantum states. In Sec. III, we calculate explicitly the spin-momentum entanglement of relativistic

quantum state. In sec. IV, we explain how we can use the quantum gate via Lorentz transformation. The last section contains the concluding remarks. The paper also contains two appendixes.

2 Single-particle relativistic quantum states

Suppose we have a bipartite system with its quantum degrees of freedom distributed among two parties $\mathcal{A}$ and $\mathcal{B}$ with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, (the standard Hilbert space of dimension d endowed with usual inner product denoted by $\langle\ .\ \rangle$). In this paper quantum state is made up of a single-particle having two types of degrees of freedom : momentum p and spin σ. The former is a continuous variable with Hilbert space of infinite dimension but we restrict ourselves here to 2D momentum subspace with two eigen-state p_1 and p_2, while the latter is a discrete one with Hilbert space of spin particle. The pure quantum state of such a system can always be written as

$$|\psi\rangle = \sum_{i=1}^{2}\sum_{j=n}^{-n} c_{ij}|p_i\rangle \otimes |j\rangle, \tag{2.1}$$

where $|p_{1(2)}\rangle$ are two momentum eigen states of each particle and the kets $|j\rangle$ are the eigenstates of spin operator. c_{ij}'s are complex coefficients such that $\sum_{i,j}|c_{ij}|^2 = 1$.

A bipartite quantum mixed state is defined as a convex combination of bipartite pure states (2.1), i.e.

$$\rho = \sum_{i=1}^{4} P_i|\psi_i\rangle\langle\psi_i|, \tag{2.2}$$

where $P_i \geq 0$, $\sum_i P_i = 1$. $|\psi_i\rangle$ $(i = 1, 2, 3, 4)$ as four orthogonal maximal entangled Bell states (BD) are belong to the product space $\mathcal{H}_A \otimes \mathcal{H}_B$ and in terms of momentum and spin states are well-known as

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|p_1\rangle \otimes |n\rangle + |p_2\rangle \otimes |-n\rangle),$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|p_1\rangle \otimes |n\rangle - |p_2\rangle \otimes |-n\rangle),$$
$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|p_2\rangle \otimes |n\rangle + |p_1\rangle \otimes |-n\rangle),$$
$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|p_2\rangle \otimes |n\rangle - |p_1\rangle \otimes |-n\rangle). \tag{2.3}$$

Here, $|\pm n\rangle$ are the Bloch sphere representation of spin state (qubit) as

$$|n\rangle = \begin{pmatrix} \cos\frac{\xi}{2} \\ e^{i\tau}\sin\frac{\xi}{2} \end{pmatrix}, |-n\rangle = \begin{pmatrix} \sin\frac{\xi}{2} \\ -e^{i\tau}\cos\frac{\xi}{2} \end{pmatrix}, \tag{2.4}$$

where ξ and τ are polar and azimuthal angles, respectively.

2.1 Relativistic single spin-$\frac{1}{2}$ particle quantum states

We assumed that spin and momentum are in the yz-plane ($\tau = \frac{\pi}{2}$, $\vec{p} = (0, p\sin\theta, p\cos\theta)$) and the Lorentz boost is orthogonal to it. For an observer in another reference frame S' described by an arbitrary boost Λ in the x-direction, the transformed BD states are given by (see Appendix A)

$$|\psi_i\rangle \longrightarrow U(\Lambda)|\psi_i\rangle,$$

$$|\Lambda\psi_1\rangle = \frac{1}{\sqrt{2}}\Bigg\{|\Lambda p_1\rangle \otimes \begin{pmatrix} \cos\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_1}}}{2} - i\sin\frac{\Omega_{\vec{p_1}}}{2}\sin\frac{\zeta}{2} \\ i\sin\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_1}}}{2} + \sin\frac{\Omega_{\vec{p_1}}}{2}\cos\frac{\zeta}{2} \end{pmatrix} + |\Lambda p_2\rangle \otimes \begin{pmatrix} \sin\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_2}}}{2} - i\sin\frac{\Omega_{\vec{p_1}}}{2}\cos\frac{\zeta}{2} \\ -i\cos\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_2}}}{2} - \sin\frac{\Omega_{\vec{p_2}}}{2}\sin\frac{\zeta}{2} \end{pmatrix}\Bigg\},$$

$$|\Lambda\psi_2\rangle = \frac{1}{\sqrt{2}}\Bigg\{|\Lambda p_1\rangle \otimes \begin{pmatrix} \cos\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_1}}}{2} - i\sin\frac{\Omega_{\vec{p_1}}}{2}\sin\frac{\zeta}{2} \\ i\sin\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_1}}}{2} + \sin\frac{\Omega_{\vec{p_1}}}{2}\cos\frac{\zeta}{2} \end{pmatrix} - |\Lambda p_2\rangle \otimes \begin{pmatrix} \sin\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_2}}}{2} - i\sin\frac{\Omega_{\vec{p_1}}}{2}\cos\frac{\zeta}{2} \\ -i\cos\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_2}}}{2} - \sin\frac{\Omega_{\vec{p_2}}}{2}\sin\frac{\zeta}{2} \end{pmatrix}\Bigg\},$$

$$|\Lambda\psi_3\rangle = \frac{1}{\sqrt{2}}\{|\Lambda p_2\rangle \otimes \begin{pmatrix} \cos\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_2}}}{2} + i\sin\frac{\Omega_{\vec{p_2}}}{2}\sin\frac{\zeta}{2} \\ i\sin\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_2}}}{2} - \sin\frac{\Omega_{\vec{p_2}}}{2}\cos\frac{\zeta}{2} \end{pmatrix} + |\Lambda p_1\rangle \otimes \begin{pmatrix} \sin\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_1}}}{2} + i\sin\frac{\Omega_{\vec{p_1}}}{2}\cos\frac{\zeta}{2} \\ -i\cos\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_1}}}{2} + \sin\frac{\Omega_{\vec{p_1}}}{2}\sin\frac{\zeta}{2} \end{pmatrix}\},$$

$$|\Lambda\psi_4\rangle = \frac{1}{\sqrt{2}}\{|\Lambda p_2\rangle \otimes \begin{pmatrix} \cos\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_2}}}{2} + i\sin\frac{\Omega_{\vec{p_2}}}{2}\sin\frac{\zeta}{2} \\ i\sin\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_2}}}{2} - \sin\frac{\Omega_{\vec{p_2}}}{2}\cos\frac{\zeta}{2} \end{pmatrix} - |\Lambda p_1\rangle \otimes \begin{pmatrix} \sin\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_1}}}{2} + i\sin\frac{\Omega_{\vec{p_1}}}{2}\cos\frac{\zeta}{2} \\ -i\cos\frac{\xi}{2}\cos\frac{\Omega_{\vec{p_1}}}{2} + \sin\frac{\Omega_{\vec{p_1}}}{2}\sin\frac{\zeta}{2} \end{pmatrix}\}, \tag{2.5}$$

where $\zeta = (\xi - 2\theta)$ and $\{|\Lambda p_1\rangle, |\Lambda p_2\rangle\}$ are two orthogonal momentum eigen-state after Lorentz transformation.

The BD density matrix (2.2), which describes the state of the single-particle at non-relativistic frame, is exchanged to the density matrix ρ' after Lorentz transformation, i.e.

$$\rho \longrightarrow U(\Lambda)\rho,$$

$$\rho' = U(\Lambda)\rho = \sum_{i=1}^{4} P_i|\Lambda\psi_i\rangle\langle\Lambda\psi_i|. \tag{2.6}$$

It can be calculate that $|\psi_i\rangle$ will be orthogonal after Lorentz transformation, i. e.

$$\langle\Lambda\psi_i|\Lambda\psi_j\rangle = \delta_{ij}.$$

3 Spin-momentum correlation

We know that a system is entangle when its density matrix cannot be written as a convex sum of product states. For a pure state, dividing the system into two subsystems, $\mathcal{A}$ and $\mathcal{B}$, allows

the Von Neumann entropy to be used as a measure of entanglement that corresponds to Ref.[6] is not Lorentz invariant. When a bipartite system is in a mixed state, there are a number of proposals for measures of the entanglement of it, including the entanglement of formation [19, 20, 21, 22], relative entropy of entanglement [23] and distillation of entanglement[24]. For pure states Each of these reduce to the von Neumann entropy. The most well-know bipartite measure of entanglement is entanglement of formation. Because of this, we apply the concurrence which is introduced by Wootters related to the entanglement of formation to measure the mixed-state entanglement of spin-momentum in the inertial frame.

3.1 Spin-momentum correlation of pure state

We show that by the von Neumann entropy the entanglement for a pure state in the Schmidt form [25] is not invariant after Lorentz transformation, and depend on the angles between spin and momentum. We introduce the following pure state

$$|\psi\rangle = \sqrt{\lambda_1}|n\rangle \otimes |p_1\rangle + \sqrt{\lambda_2}|-n\rangle \otimes |p_2\rangle, \tag{3.7}$$

where $\lambda_1 + \lambda_2 = 1$.

We take the trace over the momentum eigen states and we obtain the following reduced spin density matrix

$$\rho' = Tr_{\Lambda p_1,\Lambda p_2}(|\Lambda\psi\rangle\langle\Lambda\psi|),$$

with the following two different eigenvalues

$$\begin{aligned}
\eta_1 &= \frac{1}{2}\{\lambda_1 + \lambda_2 - \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2(\cos 2\varphi - 2\cos^2\varphi\cos(\Omega_{p_1} - \Omega_{p_2}) - 1)}\}, \\
\eta_2 &= \frac{1}{2}\{\lambda_1 + \lambda_2 + \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2(\cos 2\varphi - 2\cos^2\varphi\cos(\Omega_{p_1} - \Omega_{p_2}) - 1)}\},
\end{aligned} \tag{3.8}$$

where φ is the angle between the spin and momentum ($\varphi = \xi - \theta$). After some mathematical

manipulations we have(see Appendix B)

$$E(\rho') \leq E(\rho). \tag{3.9}$$

It shows that inequality (3.9) shows that when Lorentz boost and momentum are perpendicular, spin-momentum entanglement is decreases with increasing of velocity of the observer, as well as when spin and momentum are perpendicular, i. e.

$$\varphi = \frac{\pi}{2} \Rightarrow \eta_1 = \lambda_1, \eta_2 = \lambda_2,$$

that show Lorentz transformation does not change the entanglement between them, i.e. $E(\rho') = E(\rho)$.

3.2 Spin-momentum entanglement of mixed state

This subsection is devoted to calculate the concurrence of relativistic BD mixed state is given in (2.6). By using the Appendix A, we obtain the following result:

$$\lambda_1 = \frac{1}{2\sqrt{2}}\{\sqrt{A_1 + B_1 - \sqrt{C_1 D_1}}\},$$
$$\lambda_2 = \frac{1}{2\sqrt{2}}\{\sqrt{A_1 + B_1 + \sqrt{C_1 D_1}}\},$$
$$\lambda_3 = \frac{1}{2\sqrt{2}}\{\sqrt{A_2 + B_2 - \sqrt{C_2 D_2}}\},$$
$$\lambda_4 = \frac{1}{2\sqrt{2}}\{\sqrt{A_2 + B_2 + \sqrt{C_2 D_2}}\},$$

where

$$A_{1(2)} = 3P_{2(1)}^2 + 3P_{3(4)}^2 - (P_{2(1)}^2 + P_{3(4)}^2)\cos 2\varphi,$$
$$B_{1(2)} = 2\cos^2\varphi(2P_{2(1)}P_{3(4)} + (P_{2(1)} - P_{3(4)})^2\cos\omega),$$
$$C_{1(2)} = (P_{2(1)} - P_{3(4)})^2(-3 + \cos 2\varphi - 2\cos^2\varphi\cos\omega),$$
$$D_{1(2)} = (-(3P_{2(1)}+P_{3(4)})(P_{2(1)}+3P_{3(4)})+(P_{2(1)}-P_{3(4)})^2(\cos 2\varphi-2\cos^2\varphi\cos\omega)),$$

where λ_i's are the square roots of the eigenvalues $\rho\tilde{\rho}$ and $\omega = (\Omega_{p_1} + \Omega_{p_2})$. First index "1" in (A,B,C,D) corresponds to the (P_2, P_3) and the second index "2" corresponds to the (P_1, P_4), Therefore

$$C(\rho') = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4). \tag{3.10}$$

To see the behavior of concurrence with respect to the boost in x direction, after some calculation we obtain the following results,

$$\begin{aligned}(\lambda_{1(3)} - \lambda_{2(4)})^2 &= (P_{2(1)} - P_{3(4)})^2(1 - \cos^2\varphi \sin^2\frac{\omega}{2}),\\ (\lambda_{1(3)} + \lambda_{2(4)})^2 &= (P_{2(1)} + P_{3(4)})^2 - (P_{2(1)} - P_{3(4)})^2 \cos^2\varphi \sin^2\frac{\omega}{2}.\end{aligned} \tag{3.11}$$

Using by Eqs (3.11), we obtain

$$(\lambda_3 - \lambda_4) - (\lambda_1 + \lambda_2) = (P_1 - P_4)\sqrt{(1 - \cos^2\varphi \sin^2\frac{\omega}{2})} - \sqrt{(P_2 + P_3)^2 - (P_2 - P_3)^2 \cos^2\varphi \sin^2\frac{\omega}{2}},$$

it is easy to see that

$$(P_2 + P_3)^2 - (P_2 - P_3)^2 \cos^2\varphi \sin^2\frac{\omega}{2} \geq (P_2 + P_3)^2(1 - \cos^2\varphi \sin^2\frac{\omega}{2}),$$

so we have

$$\begin{aligned}(\lambda_3 - \lambda_4) - (\lambda_1 + \lambda_2) &\leq (P_1 - P_4)\sqrt{(1 - \cos^2\varphi \sin^2\frac{\omega}{2})} - (P_2 + P_3)\sqrt{(1 - \cos^2\varphi \sin^2\frac{\omega}{2})}\\ &= (P_1 - P_4 - P_2 - P_3)\sqrt{(1 - \cos^2\varphi \sin^2\frac{\omega}{2})} \leq (P_1 - P_4 - P_2 - P_3)\end{aligned}$$

therefore

$$C(\rho') \leq C(\rho).$$

This shows that the spin-momentum correlation in single-particle mixed quantum state is

dependent on the angle between spin and momentum. Likewise, when spin and momentum are perpendicular, i. e. $\varphi = \frac{\pi}{2}$ then the concurrence is not an observer-dependent quantity in inertial frame, namely $C(\rho') = C(\rho)$.

4 Manipulating Quantum control gates via Lorentz transformation

We explain how the Lorentz transformations can be realized as quantum control gates. To do this, we consider the pure sate of (3.7) under Lorentz transformations as

$$U(\Lambda)|\psi\rangle = \sqrt{\lambda_1}|\Lambda p_1\rangle \otimes W(n_1, p_1)|n_1\rangle + \sqrt{\lambda_2}|\Lambda p_2\rangle \otimes W(n_2, p_2)|n_2\rangle, \tag{4.12}$$

where $W(n_i, p_j)$ is Wigner rotation and the spinors are rotated by the Wigner angles. As a result, the Wigner rotation essentially behaves like a quantum control gate or controlling operator with the control quantum sates $\{|p_1\rangle, |p_2\rangle\}$ and target states $(|n_1\rangle, |n_2\rangle)$. In order to better see the quantum control gate, we assume that the reference frame S' is described by an arbitrary Lorentz boost in the x-direction and momentum and spin are parallel in z-direction, i.e. $\varphi = 0$. Then the transformed states in $2 \otimes 2$ Hilbert space are given by

$$\begin{aligned}
|p_1\rangle \otimes |\frac{1}{2}\rangle &\to \cos\frac{\Omega_{p_1}}{2}|\Lambda p_1\rangle \otimes |\frac{1}{2}\rangle + \sin\frac{\Omega_{p_1}}{2}|\Lambda p_1\rangle \otimes |-\frac{1}{2}\rangle, \\
|p_1\rangle \otimes |-\frac{1}{2}\rangle &\to -\sin\frac{\Omega_{p_1}}{2}|\Lambda p_1\rangle \otimes |\frac{1}{2}\rangle + \cos\frac{\Omega_{p_1}}{2}|\Lambda p_1\rangle \otimes |-\frac{1}{2}\rangle, \\
|p_2\rangle \otimes |\frac{1}{2}\rangle &\to \cos\frac{\Omega_{p_2}}{2}|\Lambda p_2\rangle \otimes |\frac{1}{2}\rangle + \sin\frac{\Omega_{p_2}}{2}|\Lambda p_2\rangle \otimes |-\frac{1}{2}\rangle, \\
|p_2\rangle \otimes |-\frac{1}{2}\rangle &\to -\sin\frac{\Omega_{p_2}}{2}|\Lambda p_2\rangle \otimes |\frac{1}{2}\rangle + \cos\frac{\Omega_{p_2}}{2}|\Lambda p_2\rangle \otimes |-\frac{1}{2}\rangle,
\end{aligned}$$

where Λ as matrix representation of the Lorentz transformation in the computational basis $\{|\Lambda p_1\rangle|\frac{1}{2}\rangle, |\Lambda p_1\rangle|-\frac{1}{2}\rangle, |\Lambda p_2\rangle|\frac{1}{2}\rangle, |\Lambda p_2\rangle|-\frac{1}{2}\rangle\}$ is calculated as:

$$\Lambda = \begin{pmatrix} \cos\frac{\Omega_{p_1}}{2} & \sin\frac{\Omega_{p_1}}{2} & 0 & 0 \\ -\sin\frac{\Omega_{p_1}}{2} & \cos\frac{\Omega_{p_1}}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\Omega_{p_2}}{2} & \sin\frac{\Omega_{p_2}}{2} \\ 0 & 0 & -\sin\frac{\Omega_{p_2}}{2} & \cos\frac{\Omega_{p_2}}{2} \end{pmatrix},$$

In the special case where $(\Omega_{p_2} + \Omega_{p_1}) = \pi$, we obtain

$$\cos\frac{\Omega_{p_2}}{2} = \sin\frac{\Omega_{p_1}}{2}, \quad \sin\frac{\Omega_{p_2}}{2} = \cos\frac{\Omega_{p_1}}{2}$$

and in the limit of $\Omega_{p_1} \to 0$ we get

$$\underset{\Omega_{p_1}\to 0}{Lim}\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.13}$$

We know that, the Controled-Not (CNOT) gate is a two-qubit circuit that transforms target qubit from its initial eigen-state to the opposite basis state iff the 'control' qubit is in eigen-state $|-\frac{1}{2}\rangle$. Obviously, the quantum operation (4.13) flips the spin states, when the control momentum state is $|p_2\rangle$, so the matrix representation (4.13) is similar to the Controlled-Not (CNOT) gate. This CNOT is a nonlocal operation because it can actually create a maximally entangled state from a product state or vice versa. For instance, after applying the gate(4.13)on the product state$(|p_1\rangle + |p_2\rangle) \otimes |\frac{1}{2}\rangle$, we obtain the following entangled state

$$(|p_1\rangle + |p_2\rangle) \otimes |\frac{1}{2}\rangle \to |\Lambda p_1\rangle \otimes |\frac{1}{2}\rangle + |\Lambda p_2\rangle \otimes |-\frac{1}{2}\rangle, \tag{4.14}$$

and for maximally-entangled Bell state

$$\frac{1}{\sqrt{2}}(|p_1\rangle \otimes |\frac{1}{2}\rangle + |p_2\rangle \otimes |-\frac{1}{2}\rangle) \to \frac{1}{\sqrt{2}}(|\Lambda p_1\rangle - |\Lambda p_2\rangle) \otimes |\frac{1}{2}\rangle, \tag{4.15}$$

which is a separable state.

5 Conclusions

In this paper, we have considered spin-momentum correlation of massive single spin-$\frac{1}{2}$ particle quantum states which furnish an irreducible representation of the Poincare group. Instead of the superposition of all momenta we have considered only two momenta p_1 and p_2 eigen states. We have shown that the spin-momentum correlation of relativistic single spin-$\frac{1}{2}$ particle mixed state(when the momentum is perpendicular to the boost direction) is dependent on the angle between spin and momentum and when they are parallel the measure of entanglement decreases with increasing of velocity of the observer. We have also shown that the Lorentz transformations can be realized as quantum control gates and they become like the CNOT gate in the limit where $\beta \to 1$.

APPENDIX A

Wigner representation for spin-$\frac{1}{2}$

In Ref. [26], is shown that effect of an arbitrary Lorentz transformation Λ unitarily implemented as $U(\Lambda)$ on single-particle states is

$$U(\Lambda)(|p\rangle \otimes |\sigma\rangle) = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p))(|\Lambda p\rangle \otimes |\sigma'\rangle), \tag{A-i}$$

where

$$W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p), \tag{A-ii}$$

is the Wigner rotation [7]. We will consider two reference frames in this work: one is the rest frame S and the other is the moving frame S' in which a particle whose four-momentum p in S is seen as boosted with the velocity $\vec{v}$. By setting the boost and particle moving directions in the rest frame to be $\hat{v}$ with $\hat{e}$ as the normal vector in the boost direction and $\hat{p}$, respectively,

and $\hat{n} = \hat{e} \times \hat{p}$, the Wigner representation for spin-$\frac{1}{2}$ particle is found as [12],

$$D^{\frac{1}{2}}(W(\Lambda, p) = \cos\frac{\Omega_{\vec{p}}}{2} + i\sin\frac{\Omega_{\vec{p}}}{2}(\vec{\sigma}.\hat{n}), \tag{A-iii}$$

where

$$\cos\frac{\Omega_{\vec{p}}}{2} = \frac{\cosh\frac{\alpha}{2}\cosh\frac{\delta}{2} + \sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}(\hat{e}.\hat{p})}{\sqrt{[\frac{1}{2} + \frac{1}{2}\cosh\alpha\cosh\delta + \frac{1}{2}\sinh\alpha\sinh\delta(\hat{e}.\hat{p})]}}, \tag{A-iv}$$

$$\sin\frac{\Omega_{\vec{p}}}{2}\hat{n} = \frac{\sinh\frac{\alpha}{2}\sinh\frac{\delta}{2}(\hat{e}\times\hat{p})}{\sqrt{[\frac{1}{2} + \frac{1}{2}\cosh\alpha\cosh\delta + \frac{1}{2}\sinh\alpha\sinh\delta(\hat{e}.\hat{p})]}}, \tag{A-v}$$

and

$$\cosh\alpha = \gamma = \frac{1}{\sqrt{1-\beta^2}}, \cosh\delta = \frac{E}{m}, \beta = \frac{v}{c}.$$

APPENDIX B

Entanglement of formation

Let $|\psi\rangle = \sum_{i,j=1}^{N} a_{ij} e_i \otimes e_j, \quad a_{ij} \in C$ be an two-particle pure states with normalization $\sum_{i,j=1}^{N} |a_{ij}|^2 = 1$. For this pure state the entanglement of formation E is defined as the entropy of either of the two sub-Hilbert space, i. e.

$$E(|\psi\rangle) = -Tr(\rho_1 \log_2 \rho_1) = -Tr(\rho_2 \log_2 \rho_2). \tag{A-vi}$$

where ρ_1(respectively, ρ_2) is the partial trace of $|\psi\rangle\langle\psi|$ over the first (respectively, second) Hilbert space. A given density matrix ρ on $\mathcal{H}^d \otimes \mathcal{H}^d$ has pure-state decompositions of $|\psi_i\rangle$ of the form (2.2) with probabilities P_i, The entanglement of formation for the mixed state ρ is

defined as the average entanglement of the pure states of the decomposition, minimized over all possible decompositions of ρ, i. e.

$$E(\rho) = min \sum_i P_i E(|\psi_i\rangle). \tag{A-vii}$$

In the case of n=2, (A-vi) can be written as

$$E(|\psi\rangle)|_{n=2} = H(\frac{1+\sqrt{1-C^2}}{2}), \tag{A-viii}$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is binary entropy and C is called concurrence. Thus calculation of (A-vii) can be reduced to calculate the corresponding minimum of

$$C(\rho) = min\Sigma_{b=1}^{k} p_b C(|\psi_b\rangle).$$

Wootters in [20] has shown that for a 2-qubit system entanglement of formation of a mixed state ρ can be defined as

$$E(\rho) = H(\frac{1+\sqrt{1-C^2}}{2}), \tag{A-ix}$$

by

$$C(\rho) = max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{A-x}$$

where the λ_i are the non-negative eigenvalues, in decreasing order, of the Hermitian matrix

$$R \equiv \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}},$$

and

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y),$$

where $\rho^*$is the complex conjugate of ρ when it is expressed in a fixed basis such as $|\uparrow\rangle, |\downarrow\rangle$, and σ_y is $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ on the same bases.

In order to obtain the concurrence of *BD* states (2.2) we follow the method presented by Wootters in [20]. We define subnormalized orthogonal eigenvectors $|v_i\rangle$ as

$$|v_i\rangle = \sqrt{P_i}|\psi_i\rangle, \langle v_i|v_i\rangle = P_i\delta_{ij},$$

and define $|x_i\rangle$ as $|x_i\rangle = \Sigma_{j=1}^{4} U_{ij}^{*}|v_i\rangle$ for $i = 1, 2, 3, 4$ such that

$$\langle x_i|\tilde{x_j}\rangle = (U\tau U^T)_{ij} = \lambda_i \delta_{ij},$$

$$|\tilde{x_j}\rangle = \sigma_y \otimes \sigma_y |x_j^*\rangle$$

where $\tau_{ij} = \langle v_i|\tilde{v_j}\rangle$is a symmetric but not necessarily Hermitian matrix. In construction of $|x_i\rangle$ we have considered the fact that for any symmetric matrix τ one can always find a unitary matrix U in such a way that λ_i are real and non-negative, that is, they are the square roots of eigenvalues of $\tau\tau^*$ which are the same as the eigenvalues of R. Moreover one can always find U such that λ_1 being the largest one. After some calculations we get the following values for λ_i ,

$$\lambda_1 = P_1, \lambda_2 = P_2, \lambda_3 = P_3, \lambda_4 = P_4,$$

and concurrence can be evaluated as

$$C(\rho) = (P_1 - P_2 - P_2 - P_4). \qquad \text{(A-xi)}$$

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